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A New Vortex Scheme for Viscous Flows

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The purpose of this paper is to suggest a new way to discretize the viscous term of the Navier–Stokes equations, when they are approximated by a vortex method. The idea is to approximate the vorticity by convolving it with a cutoff function. We then explicitly differentiate the cutoff function to approximate the second-order spatial derivatives in the viscous term. We prove stability for the heat equation and give error estimates for the heat and the Navier–Stokes equations.

1. INTRODUCTION

Vortex methods are numerical methods for the stimulation of incompressible flows. These methods follow particle trajectories, along which vorticity is tracked. Vortex methods are used to approximate the Euler's equations as well as the Navier–Stokes equations. Chorin [8] introduced a blob-vortex method for the two-dimensional Euler's equations. The idea of introducing blobs was to smooth the singular kernel, which connects velocity and vorticity for incompressible flows. For the two-dimensional Euler's equation vorticity is a material quantity, and therefore only particle locations are updated. Chorin extended this method to three-dimensional flows [7] using filaments, along which circulation is preserved. Later on, Beale and Majda [3, 4] and Anderson [1] extended the two-dimensional blobs to three-dimensional ones. While Beale and Majda suggested approximating spatial derivatives in Lagrangian coordinates by finite differencing, Anderson explicitly differentiates the smoothed kernel in Euleran coordinates to approximate spatial derivatives. This scheme was tested numerically [14] and was proved to be stable and convergent [2, 5].

Chorin [7–9] and Leonard [20, 21] extended vortex methods to the Navier–Stokes equations in different ways. Leonard suggested changing the core of the blobs to exactly satisfy the heat equation. However, it was proven in [17] that the core spreading technique approximates the wrong equation, rather than the Navier–Stokes equation. Chorin approximates the heat equation in the statistical sense via a random walk algorithm. Every time step each particle takes a

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Gaussianly distributed step. This process was proved [18] to converge to the heat equation, though without high accuracy. The error in the L_2 norm decays as $n^{-1/2}$, where n is the number of particles. The convergence of the random vortex methods was recently established by Long [22] and Goodman [16]. The error from the viscous term was bounded in [22] by $h(\sqrt{h/\delta})\,|\ln h|$, where h is the initial spacing and δ is a cutoff parameter. The purpose of this paper is to represent a scheme which approximates the viscous term with high accuracy.

In order to gain high-order accuracy for the viscous term we have to accurately approximate the Laplacian of the vorticity. The idea is to convolve the vorticity with a cutoff function and then approximate the second-order derivatives of the Laplacian operator by explicit differentiation of the cutoff function. In fact, other numerical methods, such as spectral and finite elements methods, can be represented in the same way (see [15].) The numerical method is therefore determined by the choice of the cutoff function and the numerical approximation of the integrals involved in the convolution. The only distinction of the method represented here from other numerical approximations is the dependence of the grid on time. In vortex methods the grid is moving with the particles and one needs to accurately approximate spatial derivatives on a time-dependent grid. For this purpose we made use of the incompressibility of the flow to approximate integrals. It was therefore possible to retain the accuracy of the integration formula, applied initially on a uniform grid. This scheme is simple to apply, retains the grid-free features of vortex methods, and is a natural extension of the non-viscous schemes. We prove the stability for the heat equation and the consistency for the heat and the Navier–Stokes equations and give error estimates. The truncation error is determined by the order of the cutoff function. One may choose the cutoff function, such that arbitrary order of convergence is obtained. We applied the scheme to the Navier–Stokes equations, once with non-smooth initial conditions and once with periodic initial conditions. The numerical results demonstrate the accuracy of the scheme, even for a relatively coarse initial grid.

Another deterministic method for the simulation of the convective diffusion equations by particle methods was proposed in [11]. This was done by replacing the diffusion operator by an integral one, and in this sense there is a similarity between the method represented in [11] and the one proposed here. It seems, though, that the approach represented here is less complicated and is easier to apply and analyze. Stability was proven in [11] for a positive kernel. It is well known that high-order kernels cannot be positive everywhere. For the stability of the scheme for the heat equation we require that the Fourier transform of the cutoff function be non-negative. This can be achieved even for an infinite-order cutoff function.

The paper is organized as follows. In Section 2 the new scheme is represented and in Sections 3 and 4 we prove the stability and the consistency of the scheme and give error estimates. In Section 5 we compare the core-spreading scheme with our scheme and in Section 6 we represent numerical results.

2. A New Scheme for Viscous Flows

The object of this paper is to give a high-order numerical approximation for the Navier–Stokes equations, using a vortex method. The Navier–Stokes equations, formulated for the vorticity ξ are given

$$\partial_t \xi + (\mathbf{u} \cdot \nabla)\xi - (\xi \cdot \nabla)\mathbf{u} = R^{-1} \Delta \xi,$$
$$\operatorname{div} \mathbf{u} = 0,$$

where $\xi = \operatorname{curl} \mathbf{u}$, $\mathbf{u} = (u, v, w)$ is the velocity vector, and $\Delta = \nabla^2$ is the Laplace operator. $R = UL/v$ is the Reynolds number, where U and L are typical velocity and length, respectively, and v is the viscosity. We follow the characteristic lines

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}, \tag{2.1}$$

along which the vorticity evolution is given by

$$\frac{d\xi}{dt} = (\xi \cdot \nabla)\mathbf{u} + R^{-1} \Delta \xi. \tag{2.2}$$

In addition, the following relation between velocity and vorticity holds for incompressible flow [10].

$$\mathbf{u}(\mathbf{x}, t) = \int K(\mathbf{x} - \mathbf{x}')\, \xi(\mathbf{x}', t)\, d\mathbf{x}'. \tag{2.3}$$

If we substitute (2.3) in (2.1), we get the system of ordinary differential equations,

$$\frac{d\mathbf{x}}{dt} = \int K(\mathbf{x} - \mathbf{x}')\, \xi(\mathbf{x}', t)\, d\mathbf{x}', \tag{2.4}$$

$$\frac{d\xi}{dt} = (\xi \cdot \nabla)\mathbf{u} + R^{-1} \Delta \xi. \tag{2.5}$$

We set an initial uniform grid $\mathbf{x}_j(0)$, $j = 1, ..., n$ with spacing h_1, h_2, h_3 for a three-dimensional problem and h_1, h_2 for a two-dimensional one. For simplicity, we assume $h_1 = h_2 = h_3 = h$. We approximate the initial vorticity by $\xi^h(\mathbf{x}, 0) = \sum_{j=1}^{n} \delta(\mathbf{x} - \mathbf{x}_j)\kappa_j^h$, where $\kappa_j^h = h^N \xi(\mathbf{x}_j, 0)$. Here $N = 2, 3$ is the dimension of the problem. Let $\mathbf{x}_j^h(t)$, $\xi_j^h(t)$ be the approximate particle locations and the approximate vorticity respectively at time t, then Eq. (2.4) is discretized by (see [7, 8])

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_{j=1}^{n} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\, \xi_j^h(t)\, h^N.$$

Here we approximate the singular kernel $K(\mathbf{x})$ by a smoothed one $K_\delta(\mathbf{x})$, where $K_\delta = \phi_\delta * K$ and $\phi_\delta(\mathbf{x}) = (1/\delta^N)\, \phi(\mathbf{x}/\delta)$. The function $\phi(\mathbf{x})$ is called a cutoff function.

The object now is to approximate the spatial derivatives appearing in (2.5). One of the terms in which spatial derivatives appear is $\xi \cdot \nabla \mathbf{u}$. This term is called the stretching term and vanishes in the two-dimensional case. For a three-dimensional problem we approximate the stretching term by explicit approximation of the smoothed kernel, as suggested in [1]. More explicitly, we approximate this term by

$$\xi_i^h(t) \sum_{j=1}^{n} \nabla_{\mathbf{x}} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\, \xi_j^h(t)\, h^N.$$

Here $\nabla_{\mathbf{x}} K_\delta$ is an explicit differentiation of the smoothed kernel in Euleran coordinates.

We now represent the approximation for the viscous term $R^{-1} \Delta\xi$. The idea is to approximate the vorticity by convolving it with a cutoff function, therefore ξ is approximated by $\phi_\delta * \xi$. We then derive an approximation to the Laplacian of the vorticity by differentiating this convolution, i.e., by $\Delta(\phi_\delta * \xi) = \Delta\phi_\delta * \xi$. Finally, we approximate the integrals involved in the convolution by the trapezoid rule and obtain

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_{j=1}^{n} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\, \xi_j^h(t)\, h^N, \tag{2.6}$$

$$\frac{d\xi_i^h(t)}{dt} = \xi_i^h(t) \cdot \sum_{j=1}^{n} \nabla_{\mathbf{x}} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\, \xi_j^h(t)\, h^N + R^{-1} \sum_{j=1}^{n} \Delta\phi_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\, \xi_j^h(t)\, h^N. \tag{2.7}$$

This yields a scheme which is similar in nature to that applied for the Euler's equations.

It is also possible to construct a similar scheme if one wishes to apply time-splitting to the Navier–Stokes equations. In this case, one may split the Navier–Stokes equations to the Euler and the heat equations. The approximation for the Euler equations is therefore

$$\begin{aligned} \frac{d\mathbf{x}_i^h(t)}{dt} &= \sum_{j=1}^{n} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\, \xi_j^h(t)\, h^N \\ \frac{d\xi_i^h(t)}{dt} &= \xi_i^h(t) \cdot \sum_{j=1}^{n} \nabla_{\mathbf{x}} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\, \xi_j^h(t)\, h^N, \end{aligned} \tag{2.8}$$

and the approximation for the heat equation is

$$\frac{\partial\xi_i^h(t)}{\partial t} = R^{-1} \sum_{j=1}^{n} \Delta\phi_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\, \xi_j^h(t)\, h^N. \tag{2.9}$$

3. STABILITY

We shall prove stability for the heat equation in two and three dimensions. For simplicity, we consider the continuous version of (2.9). The discrete one (2.9) may be treated in a similar way, using discrete Fourier transforms rather than the continuous ones. Therefore, in our proof we shall consider the case, in which the approximation for the Laplacian is given by the convolution $\Delta\phi * \xi^h$, instead of the trapezoid sum. Consider

$$\frac{\partial \xi^h(\mathbf{x}, t)}{\partial t} = R^{-1}\,\Delta\phi_\delta(\mathbf{x}) * \xi^h(\mathbf{x}, t),$$
$$\xi^h(\mathbf{x}, 0) = \xi_0(\mathbf{x}). \tag{3.1}$$

Let us define for $p \in [1, \infty)$ and $m \geqslant 0$ the Sobolev spaces

$$W^{m,p} = \{f, \partial^\alpha f \in L^p(R^n), |\alpha| \leqslant m\}$$

and by $\|\cdot\|_{m,p}$ the norm

$$\|f\|_{m,p} = \left(\sum_{0 \leqslant |\alpha| \leqslant m} \|\partial^\alpha f\|_{0,p}^p\right)^{1/p},$$

and for $p = \infty$ the usual modification.

STABILITY THEOREM. *Let $\phi \in W^{2,1}(R^N)$ and let the Fourier transform of the cutoff function be non-negative,*

$$\hat{\phi}(s) \geqslant 0, \tag{3.2}$$

then (3.1) *is stable, i.e.,*

$$\int (\xi^h(\mathbf{x}, t))^2\, d\mathbf{x} \leqslant \int (\xi^h(\mathbf{x}, 0))^2\, d\mathbf{x}.$$

Proof. Taking the Fourier transform of (3.1) yields

$$\frac{\partial \hat{\xi}^h}{\partial t}(\mathbf{s}, t) = -R^{-1}(\mathbf{s}\cdot\mathbf{s})\,\hat{\phi}_\delta(\mathbf{s})\,\hat{\xi}^h(\mathbf{s}, t).$$

Multiplying the last equality by the complex conjugate of $\hat{\xi}^h(\mathbf{s}, t)$ and integrating over $\mathbf{s}$ yields

$$\frac{\partial}{\partial t}\int |\hat{\xi}^h(\mathbf{s}, t)|^2\, d\mathbf{s} = -R^{-1}\int (\mathbf{s}\cdot\mathbf{s})\,\hat{\phi}(\delta\mathbf{s})\,|\hat{\xi}^h(\mathbf{s}, t)|^2\, d\mathbf{s}. \tag{3.3}$$

Here we also used the relation $\hat{\phi}_\delta(\mathbf{s}) = \hat{\phi}(\delta\mathbf{s})$. The right-hand side of (3.3) is non-positive by (3.2), therefore if we apply the parseval equality, we find that

$$\int (\xi^h(\mathbf{x}, t))^2\, d\mathbf{x} \leqslant \int (\xi^h(\mathbf{x}, 0))^2\, d\mathbf{x}.$$

We list several examples for which condition (3.2) is satisfied. In all of them, except Example 5, the cutoff function is radially symmetric.

EXAMPLE 1. Second-order cutoff function $\phi(r)=(1/\pi)\,e^{-r^2}$ suggested by Beale and Majda [4]. To calculate the Fourier transform, we use polar coordinates and the identity [25, 19] $\int_0^{2\pi} e^{irt\sin\phi}\,d\phi=2\pi J_0(rt)$. We also use the following property of Bessel functions [25, p. 393].

$$\int_0^\infty J_0(rs)\,re^{-p^2r^2}\,dr=\frac{1}{2p^2}\,e^{-s^2/4p^2},$$

where $J_0(s)$ is a Bessel function of order zero. We therefore find that the Fourier transform of ϕ is

$$\hat{\phi}(s)=2\int_0^\infty J_0(rs)\,re^{-r^2}\,dr=e^{-s^2/4}.$$

It is clear that with this cutoff function the method is stable.

EXAMPLE 2. Fourth-order cutoff function [4] $\phi(r)=(1/2\pi)[4e^{-r^2}-e^{-r^2/2}]$. In this case

$$\hat{\phi}(s)=2e^{-s^2/4}-e^{-s^2/2}=e^{-s^2/4}[2-e^{-s^2/4}]\geqslant 0.$$

We used this cutoff function in our numerical experiments.

EXAMPLE 3. Hald's infinite-order cutoff function [19],

$$\phi(r)=\frac{1}{3\pi r^2}\,[4J_2(2r)-J_2(r)].$$

The Fourier transform is

$$\hat{\phi}(s)=\begin{cases}1, & 0\leqslant s\leqslant 1\\ 4-s^2, & 1\leqslant s\leqslant 2\\ 0, & s\geqslant 2.\end{cases}$$

EXAMPLE 4. Another example of Hald's infinite-order cutoff function [19].

$$\phi(r)=\frac{4}{45\pi r^3}\,[16J_3(4r)-10J_3(2r)+J_3(r)].$$

$$\hat{\phi}(s)=\begin{cases}1, & 0\leqslant s\leqslant 1\\ 44+2s^2-s^4, & 1\leqslant s\leqslant 2\\ 256-32s^2+s^2, & 2\leqslant s\leqslant 4\\ 0, & s\geqslant 4.\end{cases}$$

This function is non-negative for all s.

EXAMPLE 5. For a periodic problem one may use a spectrally accurate cutoff function [15]. This function is not radially symmetric,

$$\phi(x, y)=\frac{1}{(2\pi)^2}\left[1+2\sum_{k=1}^{p}\cos kx\right]\left[1+2\sum_{l=1}^{p}\cos ly\right].$$

In this case $\hat{\phi}(k, l)=1$, for $k, l=0, \pm 1, ..., \pm p$, and otherwise $\hat{\phi}(s)=0$.

We turn now to the question of the accuracy of the scheme.

4. CONSISTENCY

CONSISTENCY THEOREM. *Let the cutoff function ϕ satisfy the conditions*

$$\phi \in W^{m+2,1}(R^N), \qquad m \geqslant 1, \tag{4.1}$$

$$\int_{R^N}\phi(\mathbf{x})\,d\mathbf{x}=1, \qquad \int_{R^N}\mathbf{x}^{\alpha}\phi(\mathbf{x})\,d\mathbf{x}=0, \quad |\alpha| \leqslant d-1, \qquad \int_{R^N}|\mathbf{x}|^{d}\,\phi(\mathbf{x})\,d\mathbf{x}<\infty. \tag{4.2}$$

Let $\mathbf{x}_j$, $j=1, ..., n$ be uniformly distributed grid points in R^N. Then, there exists a constant C such that

$$\|e_t\|_{0,2}=\left\|\Delta\xi-\sum_{j=1}^{n}\Delta\phi_\delta(\mathbf{x}-\mathbf{x}_j)\,\xi_j h^N\right\|_{0,2}\leqslant C\left(\delta^d+\frac{h^m}{\delta^{m+2}}\right).$$

Proof. We write the truncation error in (2.9) as a sum of the regularization error and the discretization one,

$$e_t=\Delta\xi-\sum_{j=1}^{n}\Delta\phi_\delta(\mathbf{x}_i-\mathbf{x}_j)\,\xi_j h^N,$$

$$e_t=e_r+e_d,$$

where

$$e_r=\Delta\xi-\Delta\phi_\delta*\xi,$$

$$e_d=\Delta\phi_\delta*\xi-\sum_{j=1}^{n}\Delta\phi_\delta(\mathbf{x}-\mathbf{x}_j)\,\xi_j h^N.$$

We approximate the regularization error by expanding its Fourier transform in Taylor series [1, 23, p. 267]. This yields

$$\|e_r\|_{0,2}=\|\Delta\xi-\Delta\phi_\delta*\xi\|_{0,2}=\|\Delta\xi-\phi_\delta*\Delta\xi\|_{0,2}.$$

Therefore, we find that

$$\|e_r\|_{0,2}\leqslant C\delta^d\|\Delta\xi\|_{d,2}.$$

This yields

$$\|e_r\|_{0,2} \leqslant C\delta^d \|\xi\|_{d+2,2}. \tag{4.3}$$

The discretization error originates from the replacement of the integral in the convolution by the trapezoidal rule. It was proven in [23, p. 262] that if $g \in W^{m,1}(R^N) \cap W^{m-1,1}(R^N)$ for $m \geqslant 3$, then

$$\left| \int g\, d\mathbf{x} - \sum_{j=1}^{n} g(\mathbf{x}_j)\, h^N \right| \leqslant Ch^m \|g\|_{m,1}.$$

Therefore, if $\xi \in W^{m,2} \cap W^{m-1,1}(R^N)$ for $m \geqslant 3$, then

$$\|e_d\|_2 \leqslant Ch^m \sum_{|\alpha| + |\beta| \leqslant m} \|\partial^\alpha \Delta\phi_\delta * \partial^\beta \xi\|_2.$$

We also apply the inequality

$$\|f * g\|_2 \leqslant \|f\|_1 \, \|g\|_2, \tag{4.4}$$

which was proven in [23, p. 267], and find

$$\|\partial^\alpha \Delta\phi_\delta * \partial^\beta \xi\|_2 \leqslant \|\partial^\alpha \Delta\phi_\delta\|_1 \, \|\partial^\beta \xi\|_2.$$

Since $\|\phi_\delta\|_{m+2,1} \leqslant C\delta^{-(m+2)}$ (see [23, p. 275]), we find

$$\|e_d\|_{0,2} \leqslant C \frac{h^m}{\delta^{m+2}}. \tag{4.5}$$

Combining (4.3) and (4.5) yields the desired result.

CONSISTENCY THEOREM FOR NAVIER–STOKES EQUATIONS. *Let the cutoff function ϕ satisfy the following conditions,*

$$\phi \in W^{m+2,1}(R^N), \qquad m \geqslant 1, \tag{4.6}$$

$$\int_{R^N} \phi(\mathbf{x})\, d\mathbf{x} = 1, \qquad \int_{R^N} \mathbf{x}^\alpha \phi(\mathbf{x})\, d\mathbf{x} = 0, \quad |\alpha| \leqslant d-1, \qquad \int_{R^N} |\mathbf{x}|^d \phi(\mathbf{x})\, d\mathbf{x} < \infty. \tag{4.7}$$

Let $\mathbf{x}_j(0)$, $j = 1, ..., n$, be uniformly distributed grid points in R^N. Then, there exists a constant C such that

$$\|e_t\|_{0,2} = \left\| \Delta\xi - \sum_{j=1}^{n} \Delta\phi_\delta(\mathbf{x} - \mathbf{x}_j)\, \xi_j h^N \right\|_{0,2} \leqslant C\left(\delta^d + \frac{h^m}{\delta^{m+2}} \right).$$

$$\|\eta_t\|_{0,2} = \left\| K * \xi - \sum_{j=1}^{n} K_\delta(\mathbf{x} - \mathbf{x}_j)\, \xi_j h^N \right\|_{0,2} \leqslant C\left(\delta^d + \frac{h^m}{\delta^{m-1}} \right).$$

Proof. The proof is similar to the one for the heat equation. The only difference is the approximation of the truncation error η_t for the locations of the particles. This is a standard estimate in vortex methods and need not be repeated here (see, for example, [1, 3]).

5. Core Spreading versus the New Algorithm

We reproduce the argument of Greengard [17] to demonstrate the difference between the core spreading technique and the new algorithm. We shall discuss the two-dimensional case with $R=1$. The core spreading changes the shape of each vortex core $\phi(\mathbf{x}, t)$ at time t. Thus, $\phi(\mathbf{x}, t) = G(\mathbf{x}, t) * \phi(\mathbf{x}, 0)$, where $G(\mathbf{x}, t) = (4\pi t)^{-1} e^{-|\mathbf{x}|^2/2t}$ is the heat kernel. It was proven in [17] that the core spreading algorithm approximates

$$\frac{\partial \xi}{\partial t} = -G(\mathbf{x}, t) * (\mathbf{u} \cdot \nabla \psi) + \Delta \xi,$$

where ψ is the transport of the initial weights $\xi(\mathbf{x}, 0) h^N$.

On the other hand, our algorithm approximates

$$\frac{d\mathbf{x}}{dt} = \int K(\mathbf{x} - \mathbf{x}')\, \xi(\mathbf{x}')\, d\mathbf{x}',$$

$$\frac{d\xi}{dt} = \Delta \xi,$$

which is equivalent to

$$\frac{\partial \xi}{\partial t} = -(\mathbf{u} \cdot \nabla) \xi + \Delta \xi.$$

Therefore, one can see that the difference between the core spreading algorithm and our scheme is the equation the vorticity approximates. In the core spreading algorithm the vorticity approximates the wrong equation, rather than the Navier–Stokes equation. Thus, vorticity is correctly diffused but incorrectly convected.

6. Numerical Results

We show numerical results for two test problems. The first one is the two-dimensional Navier–Stokes equation with non-smooth initial vorticity.

$$\xi(\mathbf{x}, 0) = \begin{cases} 1, & 0 \leqslant |\mathbf{x}| \leqslant 1, \\ 0, & |\mathbf{x}| \geqslant 1. \end{cases}$$

This problem was tested numerically by Roberts [24], using a random walk algorithm. We represent numerical results for this problem and compare them to the random walk results. In [24] several Reynolds number were tested for the Navier–Stokes equations. The range was $R = 1250, 5000,\ 20{,}000,\ 80{,}000$. The scheme was more sensitive to lower Reynolds number, therefore we show numerical results for $R = 1250$. We checked the rate of the change in the ratio of some moments of the vorticity. For this purpose, we used the functional

$$L(t) = \frac{\int_{R^2} |\mathbf{x}|^2\, \xi(\mathbf{x}, t)\, d\mathbf{x}}{\int_{R^2} \xi(\mathbf{x}, t)\, d\mathbf{x}},$$

which satisfies $L(t) = L(0) + 4t/R$. This functional was approximated by

$$A(t) = \frac{\sum_{j=1}^{n} |\mathbf{x}_j(t)|^2\, \kappa_j(t)}{\sum_{j=1}^{n} \kappa_j}.$$

Here κ_j is the intensity of the jth particle. As was suggested in [24], to eliminate the startup error, due to the approximation of the initial condition, we check the relative error

$$e(t) = \frac{|A(t) - A(0) - 4t/R|}{|A(t)|}.$$

We also smoothed the initial conditions to get more accurate results for the non-smooth solution. We assigned zero intensity to all particles $|\mathbf{x}| \geqslant 1 + \varepsilon$, where $\varepsilon = h/\sqrt{2}$. Therefore, ε is the largest distance for which vorticity is non-zero if we initially locate particles at $|\mathbf{x}| \leqslant 1$. We assigned h^2 intensity to every initial particle at $|\mathbf{x}| \leqslant 1 - \varepsilon$ and varied the intensity linearly for $1 - \varepsilon \leqslant |\mathbf{x}| \leqslant 1 + \varepsilon$. In Table I the relative error $e(t)$ is given for different time levels and compared with the random walk results. We chose the cutoff function described in Example 2 above, with the cutoff parameter $\delta = 1.8\sqrt{h}$. For this cutoff function

$$\Delta\phi_\delta(r) = \frac{1}{2\pi\delta^4}\left[16\left(\frac{r^2}{\delta^2} - 1\right) e^{-r^2/\delta^2} + \left(2 - \frac{r^2}{\delta^2}\right) e^{-r^2/2\delta^2}\right].$$

In both schemes we used initial spacing between the particles to be $h = h_1 = h_2 = 0.2$ in Tables I and II and $h = 0.1$ in Table III. We stepped the equation in time via the second-order modified Euler scheme [12, 13], for which the time step was chosen as $\Delta t = 0.2$ in Table I and $\Delta t = 0.1$ in Tables II and III. This produced a stable scheme, since we have to require $\Delta t \leqslant C\delta^2 = \tilde{C}h$ for stability.

One may notice that the error from the random walk is larger than the one from the deterministic process. It is also clear, from the numerical results shown here, that in most cases the error decrease as one refines the time step and the initial grid. Note from Tables I and III that it sometimes happen that the error in the diffusion rate of the disk might decrease in time. This may happen when the error changes sign, and therefore, since it is continuous as a function of time, it is zero at some time in between.

The second problem for which we checked the accuracy of our scheme is a periodic one. This problem served as a test problem for Chorin's finite-difference scheme for the Navier–Stokes equations [6]. The initial vorticity is given by $\xi(x, y, 0) = 2\cos(x)\cos(y)$. We performed our computations for $0 \leqslant x, y \leqslant 2\pi$. The exact solution for this problem is $\xi(x, y, t) = 2e^{-2t/R}\cos(x)\cos(y)$. We ran the scheme for $R = 1000$ and $R = 100$. The periodic boundary conditions were imposed as follows. For each computational particle we added the contributions of another eight particles, located at $(x \pm 2\pi, y)$, $(x, y \pm 2\pi)$, $(x \pm 2\pi, y \pm 2\pi)$, $(x \pm 2\pi, y \mp 2\pi)$. This is reasonable, since the further are the particles from the computational domain the smaller is their contribution. We checked the error in the discrete L_2 norm,

$$\|e\|_2^2 = \frac{1}{n} \sum_{j=1}^{n} |\xi_j^{\text{exact}} - \xi_j^{\text{comput}}|^2.$$

We chose the initial spacing between the particles to be $h = h_1 = h_2 = 2\pi/8$ in Table IV, $h = h_1 = h_2 = 2\pi/16$ in Tables V and VII, and $h = h_1 = h_2 = 2\pi/32$ in

TABLE I

$h = \Delta t = 0.2$

Time	Random walk	$e(t)$
$t = 1$	1.2E-2	3.9E-4
$t = 2$	7.2E-2	4.4E-4
$t = 3$	1.6E-1	3.4E-4
$t = 4$	2.6E-1	8.4E-5

TABLE II

$h = 0.2, \Delta t = 0.1$

Time	Random walk	$e(t)$
$t = 1$	7.3E-3	1.1E-4
$t = 2$	1.0E-2	2.1E-4
$t = 3$	5.7E-3	4.9E-4
$t = 4$	2.5E-2	7.5E-4

TABLE III

$h = \Delta t = 0.1$

Time	Random walk	$e(t)$
$t = 1$	5.8E-3	1.2E-5
$t = 2$	5.4E-3	8.9E-5
$t = 3$	1.2E-2	1.7E-5
$t = 4$	2.4E-2	1.9E-5

TABLE IV

$R=1000$, $h=2\pi/8$

Time	Heat equation	Navier–Stokes
$t=1$	1.2E-2	9.8E-2
$t=2$	2.5E-2	2.1E-1
$t=3$	3.6E-2	3.3E-1
$t=4$	4.9E-2	4.3E-1

TABLE V

$R=1000$, $h=2\pi/16$

Time	Heat equation	Navier–Stokes
$t=1$	1.9E-3	2.6E-3
$t=2$	7.8E-3	1.2E-2
$t=3$	1.7E-2	2.4E-2
$t=4$	3.1E-2	3.9E-2

Table VI. It is possible to pick a different cutoff parameter (δ_1) for the smoothing of the singular kernel in (2.6)–(2.7) or (2.8) and a different one (δ_2) for the smoothing of the vorticity by its convolution with a cutoff function ϕ_δ in (2.7) or (2.9). This was done to keep the grid less distorted as time evolves, since the locations of the particles are determined by their computed velocity. We chose $\delta_1 = 8\sqrt{h}$ and

TABLE VI

$R=1000$, $h=2\pi/32$

Time	Heat equation	Navier–Stokes
$t=1$	1.3E-4	1.5E-4
$t=2$	5.1E-4	6.0E-4
$t=3$	1.1E-3	1.3E-3
$t=4$	2.0E-3	2.4E-3

TABLE VII

$R=100$, $h=2\pi/16$

Time	Heat equation	Navier–Stokes
$t=1$	3.7E-3	3.5E-3
$t=2$	1.2E-2	1.4E-2
$t=3$	2.6E-2	3.0E-2
$t=4$	4.5E-2	5.2E-2

TABLE VIII

$R = 100$, $h = 2\pi/8$, Spectral Cutoff

Time	Heat equation	Navier–Stokes
$t=1$	5.4E-4	6.9E-4
$t=2$	2.1E-3	2.6E-3
$t=3$	4.5E-3	5.7E-3
$t=4$	7.8E-3	9.9E-3

$\delta_2 = 2\sqrt{h}$ for $R = 1000$. The time step was $\Delta t = 0.1$. Tables IV–VI refer to $R = 1000$ and Tables VII and VIII to $R = 100$. In all tables we give the error when we applied the scheme once for the heat equation and once for the Navier–Stokes equations. For the heat equation we did not have to specify δ_1.

One can learn from Tables IV–VI that the computed convergence rate of the scheme is between 3 and 4, since as one halves h, the error decreases by a factor of 8 to 16. In Table VII we show numerical results for $R = 100$. All the parameters were chosen as for $R = 1000$, except that in this case we set $\delta_2 = 4\sqrt{h}$. The cutoff coefficient may depend on the problem and in particular on the Reynolds number. To keep the discretization error independent of the Reynolds number, we have to adjust the constant C in $\delta_2 = C\sqrt{h}$. For this choice of δ_2 the discretization error is of order $h^{m/2-1}/RC^{2+m}$, therefore C has to be of order $R^{-1/(2+m)}$, where $\xi \in W^{m,2}$.

In Table VIII we represent numerical results for the periodic cutoff function given in Example 5, Section 3. This cutoff function was applied to (2.7) to evaluate the vorticity. Since the cutoff function is periodic, we did not have to add extra particles to satisfy the periodic boundary conditions for the heat equations. To update the particle locations (2.6) for the Navier–Stokes equations we used the fourth-order cutoff function of Beale and Majda, since the smoothed kernel K_δ is not periodic. For the nonperiodic kernel which connects the velocity and vorticity we did have to add extra points to update the locations of the particles in the Navier–Stokes equations. We chose $p = 4$, $h = 2\pi/8$, and $\Delta t = 1/(2\pi p)^2$, to satisfy the stability condition for periodic spectral cutoffs. Note that for the spectral cutoff we were able to achieve the same accuracy as for the fourth-order scheme with much fewer grid points. This is one of the features of the spectrally accurate cutoff functions.

5. Conclusions

Both theoretical and numerical arguments show that one may approximate the Navier–Stokes equations using vortex methods with high accuracy. For the first time we used a new formulation for the Navier–Stokes equations, in which the vorticity is tracked along particle trajectories. On the differential level, we approximated the Laplacian of the vorticity by an explicit differentiation of a cutoff function. Therefore, the proposed scheme for viscous flows is a natural extension of the non-viscous vortex schemes. In the numerical experiments, we observed a computed rate of convergence between three and four. The rate of convergence can be made as high as desired by choosing a high-order cutoff function. If we choose a cutoff function whose Fourier transform is positive, stability is assured for the heat equation.

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